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BOUNDARY INTEGRAL EQUATION FOR A CURVILINEAR BOUNDARY CRACK†

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The problem of calculating stress intensity factors for a solid with a surface crack is considered in a plane configuration. The cross-section of the solid, containing the crack, is mapped onto a domain in which the complex potentials are holomorphic everywhere, including the crack tip. It is shown that their determination reduces to solving a Muskhelishvili-type boundary integral equation. An algorithm for calculating the solution of this equation is given, together with the solutions of a range of problems for finite and infinite domains.

In [1–3] this problem was reduced to the solution of a singular integral equation of the first kind for the derivative of the displacement jump along the line of the crack. Below we present an alternative method which reduces the problem to a boundary integral equation, similar in structure to the well-known Muskhelishvili equation [4].

1. We consider a simply-connected (finite or infinite) domain D which is the transverse section of a body with a surface crack. The domain D is bounded by a piecewise-smooth contour Γ, part of which is the contour of the crack, and which is in general curvilinear (Fig. 1).

Since the stress state in the immediate vicinity of the contour corners other than the crack tip (the points A, A' and B in Fig. 1) is of no particular interest in this case, one can smooth the contour in the neighbourhoods of these points, for example by joining the smooth segments of the contour with circles of small radius r. Thus, without loss of generality, one can take the domain D to be bounded by a contour Γ that is smooth everywhere except at the crack tip.

We place the origin of coordinates at the crack tip and direct the abscissa along the tangent to the line of the crack in the direction in which it increases. The stress state in the domain D is governed [4] by two functions of a complex variable (a complex potential) $\varphi(\zeta)$ and $\psi(\zeta)$, where $\zeta = x+iy$, satisfying the equation

$$\overline{\varphi(\tau)}+\bar{\tau}\varphi'(\tau)+\psi(\tau)=\overline{f(\tau)} \quad \left(f(\tau)=i\int\limits_0^{s(\tau)}(p_x+ip_y)\,ds\right) \tag{1.1}$$

on the contour Γ, where τ is the complex coordinate of the point on the contour, s is the arc length along the contour, and p_x and p_y are components of the surface load vector.

If there is no crack, then the functions φ and ψ are analytic in the domain D, including its boundary [4], and in this case they can be found by solving, for example, the Muskhelishvili boundary integral equation [4]. In the presence of a crack its tip is a recurrence point, so that the Muskhelishvili equation is inapplicable [4]. The functions $\varphi(\zeta)$, $\psi(\zeta)$ are non-analytic at the crack tip. They can be represented in the following form [5]

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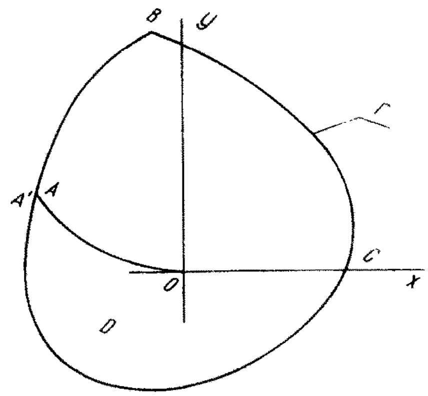

Fig. 1.

$$\varphi(\zeta)=\sqrt{\zeta}\varphi_0(\zeta)+\varphi_1(\zeta),\quad \psi(\zeta)=\sqrt{\zeta}\psi_0(\zeta)+\psi_1(\zeta) \tag{1.2}$$

where $\varphi_0(\zeta)$, $\varphi_1(\zeta)$, $\psi_0(\zeta)$, $\psi_1(\zeta)$ are analytic functions in the domain D, including the boundary contour.

We try to obtain an integral equation, similar to the Muskhelishvili equation, determining the functions φ and ψ. Let $\zeta=\omega(z)$ be a conformal map of the domain D onto the domain E such that the complex potentials φ and ψ are analytic in the domain D, including its boundary contour L. Here Eq. (1.1) transforms as [4]

$$\overline{\varphi(t)}+[\overline{\omega(t)}/\omega'(t)]\,\varphi'(t)+\psi(t)=\overline{f(t)} \tag{1.3}$$

where t is the complex coordinate of a point on the contour of E.

We look for the potentials φ and ψ in the form of products of functions analytic in the domain E (including the boundary contour)

$$\varphi(z)=\Omega(z)\,\vartheta(z),\quad \psi(z)=\Omega(z)\,\chi(z) \tag{1.4}$$

where $\Omega(z)$ is a specified function (whose form is given below), and ϑ and χ are new unknown variables. Instead of (1.3) we obtain the following relation

$$F_1(t,\bar{t})\,\overline{\vartheta(t)}+F_3(t,\bar{t})\,\vartheta(t)+F_2(t,\bar{t})\,\vartheta'(t)+\chi(t)=F_0(t,\bar{t}) \tag{1.5}$$

$$F_1(t,\bar{t})=\overline{\Omega(t)}/\Omega(t),\quad F_2(t,\bar{t})=\overline{\omega(t)}/\omega'(t)$$

$$F_3(t,\bar{t})=F_2(t,\bar{t})\,\Omega'(t)/\Omega(t),\quad F_0(t,\bar{t})=\overline{f(t)}/\Omega(t)$$

The subsequent reasoning is identical with the corresponding reasoning of Muskhelishvili. Suppose $z\notin E$. Then, expressing $\chi(t)$ using (1.5), we obtain

$$\chi(z)=\frac{1}{2\pi i}\int_L\frac{\chi(\xi)\,d\xi}{\xi-z}=\frac{1}{2\pi i}\int_L[F_0(\xi,\bar{\xi})-F_1(\xi,\bar{\xi})\,\overline{\vartheta(\xi)}- \tag{1.6}$$

$$-F_3(\xi,\bar{\xi})\,\vartheta(\xi)-F_2(\xi,\bar{\xi})\,\vartheta'(\xi)]\frac{d\xi}{\xi-z}=0$$

Suppose now that $z\to t$, remaining all the time inside the domain E. Using the Sokhotskii–Plemel formulae and the relations

$$\vartheta(t)=\frac{1}{\pi i}\int_L\frac{\vartheta(\xi)\,d\xi}{\xi-t},\quad \vartheta'(t)=\frac{1}{\pi i}\int_L\frac{\vartheta'(\xi)\,d\xi}{\xi-t} \tag{1.7}$$

valid by virtue of the analyticity of the function ϑ, after integration by parts we obtain a singular integral equation of the second kind for the function ϑ

$$F_1(t,\bar{t})\left[\overline{\vartheta(t)}+\frac{1}{2\pi i}\int_L \frac{\overline{\vartheta(\xi)\,d\xi}}{\bar{\xi}-\bar{t}}\right]+F_3(t,\bar{t})\frac{1}{2\pi i}\int_L \frac{\vartheta(\xi)\,d\xi}{\xi-t}-$$

$$-\frac{1}{2\pi i}\int_L \frac{F_1(\xi,\bar{\xi})\,\overline{\vartheta(\xi)}\,d\xi}{\xi-t}-\frac{1}{2\pi i}\int_L \frac{F_3(\xi,\bar{\xi})\,\vartheta(\xi)\,d\xi}{\xi-t}-$$

$$-\frac{1}{2\pi i}\int_L \vartheta(\xi)\,d\,\frac{F_2(t,\bar{t})-F_2(\xi,\bar{\xi})}{\xi-t}=\frac{F_0(t,\bar{t})}{2}-\frac{1}{2\pi i}\int_L \frac{F_0(\xi,\bar{\xi})\,d\xi}{\xi-t} \tag{1.8}$$

Expressing the stress intensity factors $K_{\rm I}$ and $K_{\rm II}$ in terms of the complex potential $\varphi(\zeta)$ [6] and using relation (1.4), we obtain

$$K_{\rm I}-iK_{\rm II}=2\sqrt{2\pi}\lim_{\zeta\to 0}[\zeta^{1/2}\varphi'(\zeta)]=2\sqrt{2\pi}\lim_{z\to z_0}\left\{\frac{[\omega(z)]^{1/2}}{\omega'(z)}[\Omega(z)\,\vartheta(z)]'\right\} \tag{1.9}$$

where z_0 is the coordinate of the crack tip.

2. We will show that the solution of Eq. (1.8) holds not only for points $z\notin E$, but also for points $z\in E$. We consider the function

$$\Phi(z,\bar{z})=F_2(z,\bar{z})\,\vartheta'(z)+\chi(z),\quad z\in E \tag{2.1}$$

Using the Cauchy formulae and relation (1.5) we obtain

$$\Phi(z,\bar{z})=F_2(z,\bar{z})\frac{1}{2\pi i}\int_L \frac{\vartheta'(\xi)\,d\xi}{\xi-z}+\frac{1}{2\pi i}\int_L [F_0(\xi,\bar{\xi})-F_1(\xi,\bar{\xi})\,\overline{\vartheta(\xi)}- \tag{2.2}$$

$$-F_3(\xi,\bar{\xi})\,\vartheta(\xi)-F_2(\xi,\bar{\xi})\,\vartheta'(\xi)]\frac{d\xi}{\xi-z}$$

We make z tend to the point t of the contour (here $z\in E$). Using the continuity of the function Φ and the Sokhotskii–Plemel formulae we arrive at an expression for $\Phi(t,\bar{t})$, and after substituting it into (1.5) we again obtain Eq. (1.8). Thus solutions of the equations obtained are analytically continuable in the domain E.

3. We will investigate the solvability of Eq. (1.8). Suppose first that $\Omega(z)=1$. Here, just as for the Muskhelishvili equation, Eq. (1.8) has non-trivial solutions for a zero right-hand side (eigenfunctions), or in other words, it is not in general solvable. However, if the loads applied to the body satisfy the static equations, then Eq. (1.8) by virtue of the existence theorem for solutions of problems in the theory of elasticity, has a solution, because it is identical to differential equation (1.5). The eigenfunctions of Eq. (1.8) are the result of translational displacements of the solid as a whole

$$\varphi_0=-\bar{\psi}_0=c_0 \tag{3.1}$$

and of rigid rotations

$$\varphi_1=ic_1\zeta,\quad \psi_1=0 \tag{3.2}$$

where c_0 and c_1 are arbitrary real and complex constants, respectively [7]. By the uniqueness theorem for solutions of problems in the theory of elasticity there are no other solutions [7].

The presence of eigenfunctions leads to the non-uniqueness of the solution of Eq. (1.8) when $\Omega(z)=1$. And although the eigenfunctions (3.1) and (3.2) do not affect the values of the stress intensity factors (1.9), the non-uniqueness of the solution can hinder or even make impossible the process of solving Eq. (1.8). It is with the aim of ensuring solution uniqueness that the function $\Omega(z)$is introduced into expressions (1.4). As well as being analytic this function must satisfy yet another requirement: it must vanish at every point of the domain E and of its boundary contour. Then if Eqs (3.1) hold, the eigenfunctions ϑ_0 and χ_0 are given by the expressions

$$\vartheta_0(z)=c_0\,/\,\Omega(z),\quad \overline{\chi_0(z)}=-c_0\,/\,\overline{\Omega(z)} \tag{3.3}$$

Since ϑ and χ are sought in the class of functions analytic in E, including the boundary contour, the solution (3.3) cannot be used. The introduction of the function $\Omega(z)$ thus enables one to exclude the eigenfunctions (3.1).

To eliminate the eigenfunctions (3.2) we will use a procedure similar to one employed by Sherman when transforming the Laurichella–Sherman equation [7]. We add the expression

$$b[\overline{\omega(t)}\,/\,\omega'(t)+\bar{t}\,]\,/\,\Omega(t) \tag{3.4}$$

where b is a pure imaginary constant, to the left-hand side of Eq. (1.5); we multiply both sides of the equation by $\omega'(t)\Omega(t)$ and integrate over L. Using integration by parts, we obtain

$$2i\,\mathrm{Im}\left[\int_L \overline{\Omega(t)}\,\overline{\vartheta(t)}\,\omega'(t)\,dt\right]+2ib\,\mathrm{Im}\left[\int_L \overline{\omega(t)}\,dt\right]=\int_L \overline{f(t)}\,\omega'(t)\,dt \tag{3.5}$$

If the moment of external forces vanishes, then [7]

$$\mathrm{Re}\int_L \overline{f(t)}\,\omega'(t)\,dt=0 \tag{3.6}$$

and consequently $b=0$. Thus, when the static equations (3.6) are satisfied, Eq. (1.5) and the equation modified as above are equivalent. Transforming the modified equation in the same way as Eq. (1.5), we arrive at Eq. (1.8) except that its left-hand side has the additional term

$$b\left\{[F_2(t,\bar{t})+\bar{t}\,]\,/\,[2\Omega(t)]-\frac{1}{2\pi i}\int_L[F_2(\xi,\bar{\xi})+\bar{\xi}\,]\frac{d\xi}{\Omega(\xi)\,(\xi-t)}\right\} \tag{3.7}$$

We define the constant b as follows:

$$b=i\,\mathrm{Im}\left[\frac{1}{2\pi i}\int_L\frac{\Omega(\xi)\,\vartheta(\xi)\,d\xi}{\omega(\xi)\,(\xi-z_*)}\right] \tag{3.8}$$

where z_* is some internal point of the domain E. Then, if ϑ_1 is an eigenfunction obtained from expression (3.2)

$$\vartheta_1(z)=ic_1\omega(z)\,/\,\Omega(z) \tag{3.9}$$

it then follows from (3.8) that

$$b=ic_1=0 \tag{3.10}$$

Equation (1.8) with the added term (3.7) has no non-trivial solutions when the right-hand side is zero, because here, as a consequence of (3.5), we obtain $b=0$.

4. We will determine from the above requirements the specific form of the functions $\omega(z)$ and $\Omega(z)$. We first consider the case when the domain D is finite. From formulae (1.2) it is clear that the complex potentials φ and ψ are analytic functions of the argument $\zeta^{1/2}$. Hence it is natural to take

$$\omega(z)=z^2,\quad \Omega(z)=z \tag{4.1}$$

The choice of the function Ω in this form is equivalent to the requirement that the displacement vanishes at the crack tip. Formula (1.9) then takes the form

$$K_I-iK_{II}=\sqrt{2\pi}\,\vartheta(0) \tag{4.2}$$

In order to find the coordinates of points on the contour L we write

$$\tau=Re^{i\theta},\quad t=\sqrt{\tau}=\sqrt{R}\,e^{i\theta/2} \tag{4.3}$$

For points on the contour Γ, starting at the point C (the intersection of the abscissa with the contour) and going up to the point A, and also for the upper side of the crack (the section AO), we take $\theta>0$, and on the remaining parts of the contour ($OA'C'$) $\theta<0$. As a result of the mapping (4.1) the crack tip (the point O) lies on the smooth part of the contour L.

We will consider the case when the domain D contains the point at infinity. A conformal mapping $\omega(z)$ is constructed in such a way that the domain E is finite. Such a mapping can be obtained by joining the mapping (4.1) with a mapping taking the half-plane $x>0$ inside the unit circle

$$\omega(z)=a[(z+1)/(z-1)]^2 \tag{4.4}$$

where a is a real constant

Figure 2 shows an example of a mapping (4.4): a half-plane with an oblique crack (the domain D) is mapped onto the finite domain E in which the crack tip lies on a smooth part of the boundary contour. The parameter a is taken to be the length of the crack. The sides of the crack turn into a semicircle of unit radius; the point at infinity is mapped into the point $z=1$. In order to find the coordinates t of points on the contour we need the inverse mapping

$$\tau=[(\sqrt{R}\,e^{i\theta/2}-\sqrt{a})/(\sqrt{R}\,e^{i\theta/2}+\sqrt{a})];\quad t=Re^{i\theta} \tag{4.5}$$

For points on the contour BAO (naturally, both BA and AO can in general be curvilinear), we take $\theta>0$, and for the remaining parts of the contour $\theta<0$.

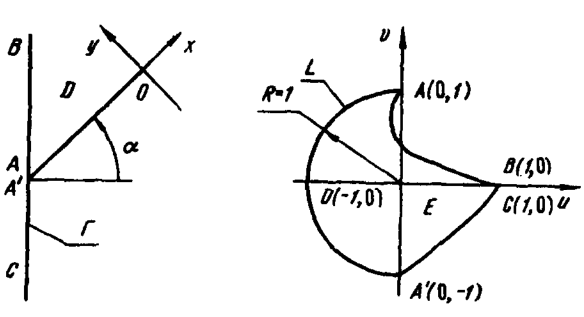

Fig. 2.

The choice of the function Ω is governed by the following conditions: one must have $\varphi=\psi=0$ at the point of infinity. Hence it is natural to put

$$\Omega=z-1 \tag{4.6}$$

Expression (1.9) takes the form

$$K_{\mathrm{I}}-iK_{\mathrm{II}}=2\sqrt{2\pi/a}\,[\vartheta(-1)-2\vartheta'(-1)] \tag{4.7}$$

With the chosen expressions (4.4) and (4.6) for the functions $\omega(z)$ and $\Omega(z)$ the eigenfunction caused by a rigid rotation (3.9) is non-analytic at the point $z=1$, i.e. it is outside the class of functions in which we look for $\vartheta(z)$. Hence when solving problems for infinite domains it is not necessary to add the term (3.4) to the left-hand side of Eq. (1.8).

5. It is natural to solve Eq. (1.8) by the boundary-element method [8]. We obtain the solution using the simplest elements, inside which the values of the function ϑ are taken to be constant. Suppose the boundary contour is decomposed into n elements. Inside each jth element we choose a collocation point t_j. Equations (1.8) and (3.8) are transformed into a system of linear complex algebraic equations for the values of the function ϑ in the elements and the constant b (below, summation over k is taken from $k=1$ to $k=n$ and integration is over the contour L_k)

$$\begin{aligned}&\bar{\vartheta}_j+\sum(A_{jk}\bar{\vartheta}_k+B_{jk}\vartheta_k)+bC_j=F_j;\quad j=1,\dots,n\\&b=i\,\mathrm{Im}\sum G_k\vartheta_k\end{aligned} \tag{5.1}$$

Separating real and imaginary parts, we obtain a system of $2n+1$ linear equations

$$\begin{aligned}&\vartheta_j^R+\sum[(A_{jk}^R+B_{jk}^R)\vartheta_k^R+(A_{jk}^{\mathrm{I}}-B_{jk}^{\mathrm{I}})\vartheta_k^{\mathrm{I}}]-b^{\mathrm{I}}C_j^{\mathrm{I}}=F_j^R\\&-\vartheta_j^{\mathrm{I}}+\sum[(A_{jk}^{\mathrm{I}}+B_{jk}^{\mathrm{I}})\vartheta_k^R+(B_{jk}^R-A_{jk}^R)\vartheta_k^{\mathrm{I}}]+b^{\mathrm{I}}C_j^R=F_j^{\mathrm{I}}\\&b^{\mathrm{I}}=\sum(G_k^{\mathrm{I}}\vartheta_k^R+G_k^R\vartheta_k^{\mathrm{I}})\end{aligned} \tag{5.2}$$

The superscripts R and I denote the real and imaginary parts of complex quantities. The coefficients of the system are given by the formulae

$$A_{jk}=\frac{1}{2\pi i}\int\frac{d\bar{\xi}}{\xi-\bar{t}_j}-\frac{1}{F_1(t_j,\bar{t}_j)}\frac{1}{2\pi i}\int\frac{F_1(\xi,\bar{\xi})\,d\xi}{\xi-t_j} \tag{5.3}$$

$$\begin{aligned}B_{jk}=&\frac{F_3(t_j,\bar{t}_j)}{F_1(t_j,\bar{t}_j)}\frac{1}{2\pi i}\int\frac{d\bar{\xi}}{\xi-t_j}-\frac{1}{F_1(t_j,\bar{t}_j)}\left\{\frac{1}{2\pi i}\int\frac{F_3(\xi,\bar{\xi})\,d\xi}{\xi-t_j}+\right.\\&\left.+\frac{1}{2\pi i}\int d\left[\frac{F_2(t_j,\bar{t}_j)-F_2(\xi,\bar{\xi})}{\xi-t_j}\right]\right\}\end{aligned} \tag{5.4}$$

$$F_j=\frac{1}{F_1(t_j,\bar{t}_j)}\left[\frac{F_0(t_j,\bar{t}_j)}{2}-\frac{1}{2\pi i}\sum\int\frac{F_0(\xi,\bar{\xi})\,d\xi}{\xi-t_j}\right] \tag{5.5}$$

$$C_j=\frac{1}{F_1(t_j,\bar{t}_j)}\left\{\frac{F_2(t_j,\bar{t}_j)+\bar{t}_j}{2\Omega(t_j)}-\frac{1}{2\pi i}\sum\int[F_2(\xi,\bar{\xi})+\bar{\xi}]\right\}\frac{d\xi}{\Omega(\xi)(\xi-t_j)} \tag{5.6}$$

$$G_k = \frac{1}{2\pi i}\int \frac{\Omega(\xi)\,\vartheta(\xi)\,d\xi}{\omega(\xi)(\xi - z_*)} \tag{5.7}$$

When $j=k$ the integrals in these formulae are singular, but by choosing functions ω and Ω in the form (4.1), (4.4) and (4.6) the conditions for the existence of principal values for these integrals are satisfied. The integration can, for example, be performed numerically [8]; when performing the calculations whose results are given below, all the above integrals with the exception of the first one in (5.3) and the second and third in (5.4), which were found exactly, were approximated by the formulae

$$I_{jk} = \frac{1}{2\pi i}\int \frac{g_k(\xi)\,d\xi}{\xi - t_j} \tag{5.8}$$

The linear law

$$g_k(\xi) = a_k\xi + b_k \tag{5.9}$$

is used to model the variation of the function g_k inside each element.

The coefficients a_k and b_k are governed by specifying values of g_k at the boundaries of the element.

Substituting (5.9) into (5.8) we obtain

$$I_{jk} = [a_k(\xi_{2k} - \xi_{1k}) + (a_k t_j + b_k)\,I]$$

$$I = \begin{cases} \ln[(\xi_{2k} - t_j)/(\xi_{1k} - t_j)], & j \neq k \\ \ln[(\xi_{2k} - t_j)/(\xi_{1k} - t_j)] + \pi i, & j = k \end{cases}$$

where ξ_{1k}, ξ_{2k} are coordinates of the initial and end points of the element.

The increment of the argument

$$\Delta\theta = \theta_2 - \theta_1; \quad r_1\exp(i\theta_1) = \xi_{1j} - t_j, \quad r_2\exp(i\theta_2) = \xi_{2j} - t_j$$

is chosen to be negative. The quantity $\vartheta(z_0)$, where z_0 is the coordinate of the crack tip, is calculated from the formula

$$\vartheta(z_0) = (\vartheta_+ + \vartheta_-)/2$$

where ϑ_+ and ϑ_- are the values of the function ϑ at elements adjacent to the crack tip on opposite sides. Such a formula is also used to find $\vartheta'(z_0)$ and the quantities ϑ'_+ and ϑ'_- are calculated using the second relation in (1.10).

Let

$$\Delta_1 = \vartheta_1 - \vartheta_n; \quad \Delta_j = \vartheta_j - \vartheta_{j-1}, \quad j = 2,\ldots,n$$

where j is the number of the element. The piecewise-constant function $\vartheta(t)$ can be represented as

$$\vartheta(t) = \vartheta_n + \sum \Delta_k H(t - \xi_{1k})$$

where H is a step function equal to zero if t belongs to the segment of the contour with a number less than k, and unity otherwise. It follows from the second formula in (1.7) that

$$\vartheta'(t) = (\pi i)^{-1} \sum \Delta_k / (\xi_{1k} - t) \tag{5.10}$$

We obtain ϑ'_+ and ϑ'_- by setting the quantity t in formula (5.10) equal to the coordinates of the collocation points in the elements adjacent to the crack tip.

6. We will consider some examples illustrating the application of the above method. Suppose that the half-plane shown in Fig. 2 extends to infinity with a uniformly distributed load of intensity σ directed parallel to the boundary of the half-plane. We shall find the dependence on the angle α of the dimensionless quantities

$$F_{\mathrm{I}} = K_{\mathrm{I}} / [\sigma(\pi a)^{1/2}], \quad F_{\mathrm{II}} = K_{\mathrm{II}} / [\sigma(\pi a)^{1/2}] \tag{6.1}$$

where a is the length of the crack. We will first deal with the case $\alpha = 0$. An analytic solution obtained by Koiter [9] gives the value 1.1215 for F_{I} ($F_{\mathrm{II}} = 0$ because of symmetry). Zang and Gudmundson obtained $F_{\mathrm{I}} = 1.121579$ [2].

We will investigate the convergence of the numerical solution to the result obtained with increasing numbers of boundary elements. The segments BA, AO, OA' and $A'C$ of the contour L were decomposed into equal numbers m of elements. The elements within the limits of each segment all had the same length. The collocation points were chosen to be the centres of the elements. The results of the calculations were

m	1	2	3	6	12	24
F_{I}	0.720	1.116	1.131	1.12289	1.12203	1.12158

We will now consider the same problem for different angles α. The principle behind decomposing the contour L into boundary elements was the same. Below we give results of the calculation for $m = 12$ ($F1$). For comparison we also give the results of Hasebe and Inohara ($F2$) which are taken from [10]

α	0°	15°	30°	45°	60°	75°
$F1_{\mathrm{I}}$	1.122	1.068	0.916	0.698	0.453	0.221
$F2_{\mathrm{I}}$	1.121	1.069	0.920	0.705	0.461	0.239
$F1_{\mathrm{II}}$	0	0.178	0.313	0.373	0.343	0.228
$F2_{\mathrm{II}}$	0	0.174	0.306	0.364	0.338	0.219

We will now consider examples of applications of the method to finite domains. Figure 3 shows a strip with a linear oblique crack of length a, stretched by a uniformly distributed load σ. The contour Γ consists of eight linear segments (the unloaded side of the strip, where there is no crack, being considered, for convenience, to be two segments). Each of these is uniformly decomposed into m boundary elements. The collocation points were chosen to be the centres of the elements. We will consider first of all the case $\alpha = 0$. Here, by symmetry, $F_{\mathrm{II}} = 0$. Let $W/H = 2$. The results of an analysis of the convergence of values of F_{I} to the value found for different values of m are shown together with the results of Bowie [11] and Zang [3] in Table 1.

Table 1

$\frac{a}{W}$	m				[11]	[3]
	1	2	6	12		
0.1	1.517	1.325	1.243	1.233	1.23	1.228
0.3	1.749	1.831	1.855	1.854	1.85	1.843
0.5	2.228	2.710	2.989	3.022	3.01	2.004
0.7	2.892	4.140	5.948	6.334	6.40	6.338

The results of calculations of F_I for the same problem with $W/H=1$, when $m=6$ and $m=12$, together with results from [12, 13], obtained for an infinite strip, are given below

a/W	0.1	0.3	0.5	0.7
$F_I(m=6)$	1.201	1.651	2.750	5.646
$F_I(m=12)$	1.192	1.658	2.811	6.194
F_I [12]	-	1.655	2.827	6.376
F_I [13]	1.189	1.660	2.825	6.36

The results of calculations of $F_I(\alpha)$ and $F_{II}(\alpha)$ when $W/H=1$ and $a/W=0.3$ (taking $m=6$) are almost identical with the results in [11].

Note that the test problem solutions obtained show that the accuracy of the method is acceptable.

In conclusion we consider the problem of a curvilinear crack in a stretched strip (Fig. 4). Suppose that the line of the crack is part of a circle of radius R. The length of the crack is $a=R\alpha$. The decomposition of the contour L into elements is the same as in the preceding problem with a linear crack in a strip. Below we give the results of calculations for $R/W=0.3$ $(m=6)$.

α	15°	30°	45°	60°	75°	90°
F_I	1.146	1.116	1.044	0.905	0.708	0.481
F_{II}	0.171	0.337	0.484	0.584	0.615	0.570

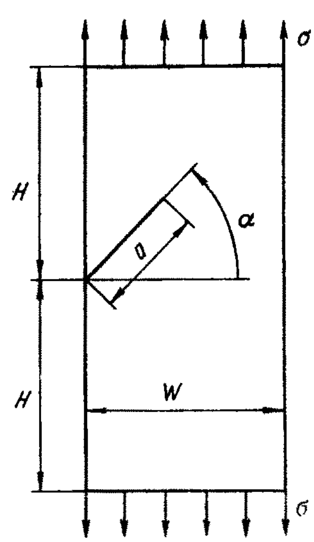

Fig. 3.

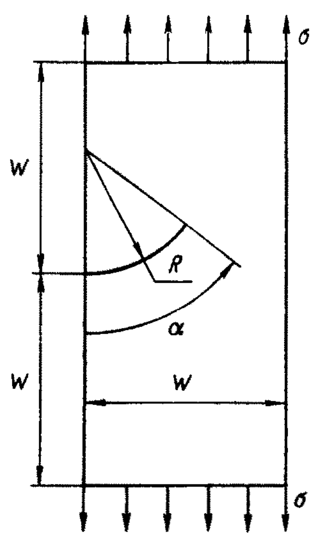

Fig. 4.

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